

A Farey tale for $\mathcal{N} = 4$ dyons

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JHEP09(2009)022

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A Farey tale for $\mathcal{N} = 4$ dyons

Sameer Murthy and Boris Pioline

*Laboratoire de Physique Théorique et Hautes Energies (LP THE),
Université Pierre et Marie Curie-Paris 6; CNRS UMR 7589,
Tour 24-25, 5^{ème} étage, Boite 126, 4 Place Jussieu,
75252 Paris Cedex 05, France*

E-mail: smurthy@lpthe.jussieu.fr, pioline@lpthe.jussieu.fr

ABSTRACT: We study exponentially suppressed contributions to the degeneracies of extremal black holes. Within Sen’s quantum entropy function framework and focusing on extremal black holes with an intermediate AdS_3 region, we identify an infinite family of semi-classical AdS_2 geometries which can contribute effects of order $\exp(S_0/c)$, where S_0 is the Bekenstein-Hawking-Wald entropy and c is an integer greater than one. These solutions lift to the extremal limit of the $SL(2, \mathbb{Z})$ family of BTZ black holes familiar from the “black hole Farey tail”. We test this understanding in $\mathcal{N} = 4$ string vacua, where exact dyon degeneracies are known to be given by Fourier coefficients of Siegel modular forms. We relate the sum over poles in the Siegel upper half plane to the Farey tail expansion, and derive a “Farey tale” expansion for the dyon partition function. Mathematically, this provides a (formal) lift from Hilbert modular forms to Siegel modular forms with a pole at the diagonal divisor.

KEYWORDS: Black Holes in String Theory, Discrete and Finite Symmetries, Black Holes, String Duality

ARXIV EPRINT: [0904.4253](https://arxiv.org/abs/0904.4253)

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1 Introduction and summary

Defining the thermodynamic entropy of a black hole in a quantum theory of gravity — even in principle — is an interesting open problem. Any candidate formula should take into account the quantum fluctuations of matter and gravity and reduce to the Bekenstein-Hawking formula in the classical, large horizon area limit. Ultraviolet fluctuations can be incorporated by applying Wald’s generalization of the Bekenstein-Hawking formula using the Wilsonian quantum effective action, but infrared fluctuations need an extra prescription. Such a proposal, called the *quantum entropy function*, has been put forward recently [1] for the case of charged extremal black holes in any number of dimensions.

This proposal relies on the near-horizon geometry of an extremal black hole being AdS_2 (or rather, a patch of global AdS_2 known as the AdS_2 black hole [2, 3]) times a compact manifold M . The quantum entropy function $d(q_i)$ is a Euclidean path integral over asymptotically AdS_2 field configurations with fixed electric charge q_i , fixed value of the scalar fields at infinity, and a Wilson line insertion. The functional integral runs over all fields in the dimensionally reduced two-dimensional field theory. The proposal comes with a specific prescription for dealing with the infrared divergence due to the infinite volume of AdS space. The ultraviolet divergences of Einstein gravity in $d \geq 3$ are

assumed to be resolved by some ultraviolet completion such as string theory; they translate into the existence of an infinite number of massive fields in two dimensions, irrelevant at low energies.

The Euclidean path integral is dominated by the field configuration corresponding to pure AdS_2 , but there are in general other saddle points approaching AdS_2 asymptotically, and leading to exponentially suppressed contributions. These saddle points do not necessarily correspond to smooth geometries, but may include (e.g. orbifold) singularities allowed by the UV completion. A proposal for including such orbifolds has been made in [4, 5].

In this work, we explain and refine this construction in the case where the AdS_2 black hole is the “very near horizon” limit [2, 3] of a BTZ black hole in AdS_3 , which could itself be embedded in a larger asymptotically flat space. We further test this understanding in the case of $\mathcal{N} = 4$ dyons in 4 dimensions, where the exact degeneracies are captured by a certain Siegel modular form. We compute the contributions to the dyon degeneracies from arbitrary poles in the Siegel upper half plane, and find agreement with the classical action of the $SL(2, \mathbb{Z})$ family of AdS_3 black holes. Finally, we relate the sum over poles to the Farey tail expansion of a certain Jacobi form, which should arise as the modified elliptic genus of the superconformal field theory dual to $\mathcal{N} = 4$ dyons.

The existence of an intermediate AdS_3 region requires that the black hole arises as a black string winding around an extra circle. This includes most of the examples in string theory where the microscopic origin of the black hole entropy has been understood [6, 7]. In this case the geometry which dominates the Euclidean path integral asymptotes to $AdS_2 \times S_1 \times \tilde{M}$ with \tilde{M} a compact manifold. The circle S_1 is non-trivially fibered over AdS_2 so as to produce a constant field strength after Kaluza-Klein reduction. An infinite family of saddle points, labelled by two relatively prime integers (c, d) with $1 \leq d < c$ can then be constructed as follows: consider an $\mathbb{Z}/c\mathbb{Z}$ orbifold of the dominant saddle point, where the cyclic generator acts as a $2\pi/c$ rotation in Euclidean AdS_2 , accompanied by a translation of angle $2\pi d/c$ along the circle S^1 . When $c > 1$ and $1 \leq d < c$, the resulting geometry is smooth, and gives a subleading contribution of order

$$\exp\left(\frac{\mathcal{S}_0}{c} + 2\pi i q \frac{d}{c}\right) \tag{1.1}$$

to the quantum entropy function, where \mathcal{S}_0 is the contribution of the dominant configuration with $(c, d) = (1, 0)$, equal to the Bekenstein-Hawking-Wald macroscopic entropy, and q is the momentum on the circle S^1 .

In the language of the parent AdS_3 , these geometries are the extremal limit¹ of the $\Gamma_\infty \backslash SL(2, \mathbb{Z})/\Gamma_\infty$ family of AdS_3 black holes discussed in [2, 8, 9]. The geometry of thermal AdS_3 is a solid torus, and the various solutions correspond to all possible ways of filling in the boundary torus with a three-dimensional smooth manifold of constant negative

¹One has to be careful in taking this limit. The boundary torus of Euclidean AdS_3 has a complex structure modulus τ . Upon zooming into the very-near-horizon region of the Euclideanized extremal BTZ black hole, one gets another torus with complex structure $(\tau \rightarrow 0, \bar{\tau} \rightarrow \infty)$, which does not admit any real sections. We discuss this in section § 3.

curvature. The two integers (c, d) label the unique cycle of the boundary torus which becomes contractible in the bulk.

In the present case, the mass (or angular momentum) of the BTZ black hole is fixed. The family of AdS_3 solutions that contribute to the entropy function therefore have asymptotic complex structure varying as a function of (c, d) . From the point of view of the microscopic theory, the AdS_3 path integral corresponds to the canonical partition function keeping the electric potential fixed. The AdS_2 path integral corresponds to the microcanonical ensemble with fixed electric charge. Going from one to the other involves summing over states with different charges, or summing over different boundary conditions; the former is the original AdS_3 Farey tail, and the latter is what we discuss in this paper.

Subleading corrections of order (1.1) have been encountered in a recent analysis of the exact microscopic degeneracies of $\mathcal{N} = 4$ dyons [4]. There is now overwhelming evidence that the latter are encoded as Fourier coefficients of certain Siegel modular forms [10–17]. The saddle points in the semi-classical expansion of the Fourier coefficients at large charges are labelled by five integers (m_1, m_2, j, n_1, n_2) , transforming linearly as the 5-dimensional representation of $Sp(2, \mathbb{Z})$. For $n_2 = 1$, the saddle point contribution $\exp(\mathcal{S}_0)$ reproduces the Bekenstein-Hawking-Wald entropy \mathcal{S}_0 , including R^2 -type quantum corrections to the four-dimensional low energy effective action. For $n_2 > 1$, the saddle point contributes a subleading correction of order [4]

$$\exp \left[\frac{\mathcal{S}_0}{n_2} + \frac{2\pi i}{n_2} \left(n_1 \frac{Q^2}{2} - \frac{j}{2} (P \cdot Q) - m^1 \frac{P^2}{2} \right) \right], \tag{1.2}$$

precisely² of the form (1.1). Moreover, the partition function $\mathcal{Z}_m(\rho, v)$ at fixed magnetic charge $P^2/2$ and fixed potentials (ρ, v) conjugate to $(Q^2/2, P \cdot Q)$ can be obtained by summing over all poles with $0 \leq m^1 < n_2$. This provides a Poincaré series representation of the Jacobi form $\mathcal{Z}_m(\rho, v)$ which is very similar to the Farey tail expansion, and hints at some intriguing relation between the Fourier coefficients of the elliptic genus of K3 and those of the Dedekind function. At any rate, this Farey tail-type expansion supports the existence of an effective black string description for any charges, and therefore the existence of an intermediate AdS_3 region.

The rest of this note is organized as follows. In section 2, we review the quantum entropy formalism of [1]. In section 3, focusing on extremal black holes with an intermediate AdS_3 region we construct an infinite family of solutions which are asymptotic to AdS_2 , and lift to the extremal limit of the $\Gamma_\infty \backslash \Gamma / \Gamma_\infty$ family of AdS_3 black holes familiar from the black hole “Farey tail”. In section 4, we proceed to analyze the exponentially suppressed corrections to degeneracies of $\mathcal{N} = 4$ dyons, and derive a “Farey tale” representation of the black hole partition function as a sum over poles in the Siegel upper half plane. The mathematically oriented reader may skip directly to section 4.

²In the $D1 - D5 - P - KKM$ duality frame, the quadratic combinations become $Q^2/2 = q, P \cdot Q = l, P^2/2 = Q_1 Q_5$, where q is the momentum along the circle S^1 discussed above and l is the momentum around a different circle inside \tilde{M} .

2 Review of the quantum entropy function formalism

The quantum entropy function [1] generalizes the Wald entropy formula to include non-local quantum corrections in a consistent quantum theory of gravity such as string theory. It is formulated for extremal black holes, whose near horizon geometry is $AdS_2 \times M$ where M is a compact manifold. The higher dimensional theory is written as a two-dimensional theory with a generally infinite set of fields, including the 2D metric, gauge fields A^i with field strengths F^i and matter fields ϕ_a governing the shape of M . The magnetic charges in higher dimension appear as fluxes on M , and generate a potential for the scalars ϕ_a as well as theta-angle couplings for the field strengths F^i . There are also higher-derivative contributions to the two-dimensional Lagrangian, induced by ultraviolet fluctuations in higher dimensions above the Wilsonian cut-off.

The most general near horizon field configuration consistent with the $SL(2, \mathbb{R})$ symmetry of AdS_2 is:

$$ds^2 = v \left[-(r^2 - 1)du^2 + \frac{dr^2}{r^2 - 1} \right], \quad F^i = e^i dr \wedge du, \quad \phi^a(u, r) = \phi_0^a. \quad (2.1)$$

where v, e^i and ϕ_0^a are constants. This is the metric of an AdS_2 black hole [2, 3] with horizon at $r = 1$. It is locally isometric to AdS_2 and the region $r > 1$ covers a triangular wedge extending halfway from the boundary into global AdS_2 [3].

An analytic continuation $u \rightarrow -iu_E$ leads to the Euclidean metric

$$ds^2 = v \left[(r^2 - 1)du_E^2 + \frac{dr^2}{r^2 - 1} \right], \quad F^i = -i e^i dr \wedge du_E, \quad \phi^a(u_E, r) = \phi_0^a. \quad (2.2)$$

This metric is non-singular at the erstwhile horizon $r^2 = 1$ provided the Euclidean time coordinate u_E is periodic modulo 2π . In the gauge $A_r^i = 0$, the gauge fields are given by

$$A^i = -i e^i (r - 1) du_E, \quad (2.3)$$

where the constant term ensures that the Wilson line $\oint_{S^1} A^i$ around the thermal circle vanishes at the horizon $r = 1$. This is needed for regularity since the thermal circle contracts to zero size.

The quantum entropy function is defined as a functional integral over all field configurations which asymptote to the AdS_2 Euclidean black hole (2.2). Specifically, one requires the fall-off conditions [18]

$$ds_0^2 = v \left[(r^2 + \mathcal{O}(1)) du_E^2 + \frac{dr^2}{r^2 + \mathcal{O}(1)} \right], \quad (2.4)$$

$$\phi^a = \phi_0^a + \mathcal{O}(1/r), \quad A^i = -i e^i (r - \mathcal{O}(1)) du_E,$$

which are invariant under an action of the Virasoro algebra. In particular, in contrast to higher dimensional instances of the AdS/CFT correspondence, the mode of the gauge field corresponding to the electric charge grows linearly (non-normalizable) and must be kept fixed, while the mode corresponding to the electric potential is constant (normalizable), and allowed to fluctuate.

The quantum entropy function, a function of the electric charges q_i and moduli ϕ_0^a , is then defined as the functional integral

$$\Omega(q_i, \phi_0^a) = \left\langle \exp \left[-i q_i \oint_{u_E} A^i \right] \right\rangle_{\text{AdS}_2}^{\text{finite}}. \quad (2.5)$$

The superscript refers to the following prescription for regulating the divergence due to the infinite volume of the AdS_2 : First, one enforces a cutoff at a large $r = r_0$ (more general cut-offs have been recently discussed in [5]). As $r_0 \rightarrow \infty$, the proper length $L \sim 2\pi\sqrt{v}r_0$ of the boundary goes to infinity, and the integrand always scales as $e^{C_1 r_0 + C_0 + \mathcal{O}(r_0^{-1})}$. The finite part is defined as e^{C_0} .

In the classical limit, the functional integral (2.5) is dominated by the saddle point where all fields take their classical values (2.2). In this case, the path integral reduces to

$$\left\langle \exp \left[-i q_i \oint_{u_E} A^i \right] \right\rangle \sim \exp \left(-S_{\text{bulk}} - S_{\text{bdry}} - i q_i \oint_{u_E} A^i \right), \quad (2.6)$$

where

$$S_{\text{bulk}} = \oint_{u_E} (r_0 - 1) v \mathcal{L} du_E \quad (2.7)$$

is the regulated two-dimensional action. Since the integrand is independent of u_E , the integral simply produces a factor of 2π . S_{bdry} has a divergent part proportional to r_0 and no constant part. The divergent part can be removed by adding an appropriate counterterm in the boundary action, leaving

$$\Omega(q_i, \phi_0^a) \sim e^{2\pi(q_i e^i - v\mathcal{L})} \equiv e^{S_0}, \quad (2.8)$$

where the electric field e^i is related to the charge q_i via

$$q_i = \frac{\partial(v\mathcal{L})}{\partial e^i}. \quad (2.9)$$

As shown in [19], the classical action S_0 reproduces the Bekenstein-Hawking-Wald entropy of the extremal black hole.

Quantum corrections to the classical answer (2.8) are of two types: (i) fluctuations around the classical field configuration (2.2), which produce power law corrections, and (ii) non-perturbative effects come from different classical solutions with the same asymptotics as (2.2) (and fluctuations about those configurations), which are exponentially suppressed with respect to (2.8).

In [4, 5], it was proposed that there is a universal series of non-perturbative corrections to the degeneracy of the form $e^{S_0/c}$ for c integer coming from orbifolds of $AdS_2 \times M$. We shall see in the next section that this is indeed borne out for BTZ black holes.

3 Subleading saddle points for extremal BTZ black holes

In this section, we construct an infinite family of solutions asymptotic to extremal BTZ black holes, and find that they lead to exponentially suppressed contributions of order (1.1) to the quantum entropy function.

3.1 The BTZ black hole

The general solution of three-dimensional gravity with scalar curvature $-6/\ell^2$ asymptotic to AdS_3 is given by the two-parameter family of BTZ black holes [20],

$$ds_3^2 = -\frac{(\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)}{\rho^2} dt^2 + \frac{\ell^2 \rho^2}{(\rho^2 - \rho_+^2)(\rho^2 - \rho_-^2)} d\rho^2 + \rho^2 \left(dy - \frac{\rho_+ \rho_-}{\rho^2} dt \right)^2, \quad (3.1)$$

where the azimuthal coordinate of AdS_3 at infinity y has periodicity 2π . The parameters $\rho_+ > \rho_- > 0$ denote the location of the outer and inner horizon, and depend on the mass M and angular momentum J via $M = (\rho_+^2 + \rho_-^2)/(8G\ell^2)$, $J = \rho_+ \rho_-/(4G\ell)$; henceforth we set $8G = 1$.

The solution (3.1) is well-known to be an orbifold of AdS_3 [21, 22]: to see this, define for the exterior region $\rho > \rho_+$,

$$z \equiv \sqrt{\frac{\rho_+^2 - \rho_-^2}{\rho^2 - \rho_-^2}} e^{(\rho_+ y - \rho_- t)/\ell}, \quad w_{\pm} \equiv \sqrt{\frac{\rho^2 - \rho_+^2}{\rho^2 - \rho_-^2}} e^{(\rho_+ \mp \rho_-)(y \pm t)/\ell} \quad (3.2)$$

The coordinates (w_+, w_-, z) parametrize an element g of $G = \text{SL}(2, \mathbb{R})$,

$$g = \begin{pmatrix} 1 & w_+ \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ w_- & 1 \end{pmatrix} \quad (3.3)$$

and the metric (3.1) is locally the bi-invariant metric on the group manifold G , $ds_3^2 = \ell^2(dz^2 + dw_+ dw_-)/z^2$. Globally, the periodicity $y \sim y + 2\pi$ implies the identification $g \sim g_L \cdot g \cdot g_R$ where

$$g_L = \begin{pmatrix} e^{\pi(\rho_+ - \rho_-)/\ell} & 0 \\ 0 & e^{-\pi(\rho_+ - \rho_-)/\ell} \end{pmatrix}, \quad g_R = \begin{pmatrix} e^{\pi(\rho_+ + \rho_-)/\ell} & 0 \\ 0 & e^{-\pi(\rho_+ + \rho_-)/\ell} \end{pmatrix} \quad (3.4)$$

are two hyperbolic elements in G .

The Euclidean section is obtained by analytically continuing both the time coordinate $t \rightarrow -it_E$ and the parameter $\rho_- \rightarrow ir_-$, and letting t_E, r_- be real. Regularity of the Euclidean section at ρ_+ requires identifying

$$(t, y) \sim \left(t + \frac{i}{T}, y + \frac{i\Omega}{T} \right), \quad T = \frac{\rho_+^2 - \rho_-^2}{2\pi\ell\rho_+}, \quad \Omega = \frac{\rho_-}{\rho_+}, \quad (3.5)$$

which amounts to the trivial identification

$$(w_+, w_-, z) \sim (e^{2\pi i} w_+, e^{-2\pi i} w_-, z) \quad (3.6)$$

on the group manifold G . The Euclidean section is a two-dimensional solid torus filled with an hyperbolic metric. The A-cycle $(t(s), y(s)) = (t_0 + is/T, y_0 + i\Omega s/T)$ with $0 \leq s < 1$ is contractible in the full geometry, hence identified as the thermal circle, while the B-cycle $(t(s), y(s)) = (t_0, y_0 + 2\pi s)$ is non-contractible. The complex structure of the torus T^2 generated by $\partial_{t_E}, \partial_y$ at fixed radial distance ρ is parametrized by the modulus

$$\tau_+ = \frac{i}{\ell} \left(\rho_- + \rho_+ \sqrt{\frac{\rho^2 - \rho_-^2}{\rho^2 - \rho_+^2}} \right), \quad (3.7)$$

We define

$$\tau_- = \frac{i}{\ell} \left(\rho_- - \rho_+ \sqrt{\frac{\rho_-^2 - \rho_+^2}{\rho_-^2 - \rho_+^2}} \right). \quad (3.8)$$

such that τ_+ and τ_- are complex conjugate to each other when ρ_- (hence the angular momentum) is imaginary. At large radius, the complex structure of the induced metric on the torus goes to a constant,

$$\tau_{\pm}^{\infty} = \frac{i}{\ell} (\rho_- \pm \rho_+). \quad (3.9)$$

3.2 The extremal limit

The extremal limit corresponds to $\rho_+ \rightarrow \rho_-$ or $\ell M \rightarrow J$, such that the temperature T goes to zero. Before taking the limit, it is convenient to change coordinates to

$$r \equiv \frac{2\rho^2 - \rho_+^2 - \rho_-^2}{\rho_+^2 - \rho_-^2}, \quad u \equiv \frac{1}{\ell}(\rho_+ - \rho_-)(t + y), \quad \phi \equiv y - \frac{\rho_-}{\rho_+}t, \quad (3.10)$$

such that the group element (3.3) is now parametrized by

$$z = \sqrt{\frac{2}{r+1}} e^{R\phi/2}, \quad w_+ = \sqrt{\frac{r-1}{r+1}} e^u, \quad w_- = \sqrt{\frac{r-1}{r+1}} e^{R\phi-u}, \quad (3.11)$$

where $R \equiv 2\rho_+/\ell$. In these coordinates, the metric (3.1) takes the form

$$ds_3^2 = \frac{\ell^2}{4} \left[-(r^2 - 1)du^2 + \frac{dr^2}{r^2 - 1} + R^2 \left(d\phi + \frac{1}{R}(r-1)du \right)^2 \right], \quad (3.12)$$

while the thermal and angular identifications are, respectively,

$$(u, \phi) \sim (u + 2\pi i, \phi) \sim \left(u + \frac{2\pi}{\ell}(\rho_+ - \rho_-), \phi + 2\pi \right). \quad (3.13)$$

We now take the extremal limit $\rho_+ \rightarrow \rho_-$, keeping the coordinates (r, u, ϕ) and parameter R fixed. The metric stays as in (3.12), but the thermal and angular identifications simplify to

$$(u, \phi) \sim (u + 2\pi i, \phi) \sim (u, \phi + 2\pi). \quad (3.14)$$

To leading order in $\lambda \equiv (\rho_+ - \rho_-)/2 \rightarrow 0$, the change of variable (3.10) coincides with the one considered in [19, 23]

$$\rho = \rho_+ + \lambda(r-1), \quad t = \frac{\ell}{4\lambda}u, \quad y = \phi + \frac{\ell}{4\lambda} \left(1 - \frac{2\lambda}{\rho_+} \right) u. \quad (3.15)$$

In the extremal limit, the complex structure (3.7),(3.8) of the (u, ϕ) torus reduces to

$$\tau_{\pm} = \frac{iR}{2} \left(1 \pm \sqrt{\frac{r+1}{r-1}} \right). \quad (3.16)$$

It is a characteristic feature of the near-horizon geometry that $\tau_+ \sim iR$ goes to a finite value while $-1/\tau_- \sim -2ir/R$ diverges linearly as $r \rightarrow \infty$. More invariantly, the left-moving complex structure τ_+ is regular at $r = \infty$ while the right-moving one τ_- reaches a cusp of the moduli space $\mathcal{H}/\text{SL}(2, \mathbb{Z})$. This is possible because τ_+ and τ_- are not complex conjugate to each other in the Lorentzian geometry, and ρ_- needs to stay real for the extremal limit to exist.

3.3 A family of extremal solutions

Given a complex structure on T^2 labelled by τ^∞ or one of its images $(a\tau^\infty + b)/(c\tau^\infty + d)$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$, there exists a unique hyperbolic metric with T^2 as boundary. There is however, an infinite family of physically distinct smooth solutions differing by the homology class of the contractible cycle in the full geometry [2, 24].

It can be parameterized by two coprime integers (c, d) as follows: given the parameters ρ_+, ρ_- of the original metric (3.1), and the corresponding asymptotic complex structure moduli (3.9), we define transformed parameters ρ'_+, ρ'_- via

$$\frac{i}{\ell}(\rho'_- \pm \rho'_+) \equiv \frac{1}{d + c/\tau_\mp^\infty}, \quad (3.17)$$

and coordinates ρ', y', t' via

$$y' \pm t' \equiv -(b + a/\tau_\mp^\infty)(y \pm t), \quad \frac{\rho'_+{}^2 - \rho'_-{}^2}{\rho'^2 - \rho'^2} \equiv \frac{\rho_+^2 - \rho_-^2}{\rho^2 - \rho^2}. \quad (3.18)$$

The solution labelled by (c, d) is obtained by replacing all quantities in the BTZ solution (3.1) by corresponding primed quantities. Thus, the metric $ds_3'^2 = \ell^2(dz'^2 + dw'_+ dw'_-)/z'^2$ is still locally isometric to AdS_3 , with

$$w'_\pm \equiv \sqrt{\frac{\rho'^2 - \rho_\pm'^2}{\rho'^2 - \rho_-'^2}} e^{(\rho'_+ \mp \rho'_-)(y' \pm t')/\ell} = \sqrt{\frac{\rho^2 - \rho_\pm^2}{\rho^2 - \rho_-^2}} \exp\left(\pm i(y \pm t) \frac{b + a/\tau_\mp^\infty}{d + c/\tau_\mp^\infty}\right) \quad (3.19)$$

and similarly for z' . The original solution (3.1) is recovered for $(c, d) = (1, 0)$, while $(c, d) = (0, 1)$ reproduces global AdS_3 with $\rho'_+ = 0$. All these solutions have the same asymptotics

$$ds_3'^2 = \rho'^2(-dt'^2 + dy'^2) + \ell^2 \frac{d\rho'^2}{\rho'^2} \quad \text{as } \rho' \rightarrow \infty, \quad (3.20)$$

and coordinate periodicities

$$(t', y') \sim \left(t' + \frac{i}{T}, y' + \frac{i\Omega}{T}\right) \sim (t', y' + 2\pi), \quad (3.21)$$

where T, Ω are the temperature and angular velocity of the original solution (3.5). However they differ in the homology of the thermal circle (i.e. the one which is contractible in the bulk). The latter is obtained by demanding that the argument of w'_\pm in (3.19) varies by $\pm 2\pi i$ as in (3.6), i.e.

$$(t'(s), y'(s)) = \left(t'_0 + \frac{ic s}{T}; y'_0 + \frac{ic s \Omega}{T} + 2\pi d s\right) \quad (3.22)$$

with $0 \leq s \leq 1$. Changing $d \rightarrow d + c$ does not affect the contractible cycle, so inequivalent solutions are labelled by double cosets $\Gamma_\infty \backslash \text{SL}(2, \mathbb{Z}) / \Gamma_\infty$ where $\Gamma_\infty = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

We now take the extremal limit of these solutions by taking $\rho'_+ \rightarrow \rho'_-$ and zooming in the region $\rho' \sim \rho'_+$. We do this as before by changing coordinates

$$r' \equiv \frac{2\rho'^2 - \rho_+'^2 - \rho_-'^2}{\rho_+'^2 - \rho_-'^2}, \quad u' \equiv \frac{1}{\ell}(\rho'_+ - \rho'_-)(t' + y'), \quad \phi' \equiv y' - \frac{\rho_-'}{\rho'_+} t', \quad (3.23)$$

and taking the above limits keeping (r', u', ϕ') fixed. In these coordinates, the metric becomes

$$ds_3'^2 = \frac{\ell^2}{4} \left[-(r'^2 - 1)du'^2 + \frac{dr'^2}{r'^2 - 1} + R'^2 \left(d\phi' + \frac{1}{R'}(r' - 1)du' \right)^2 \right], \quad (3.24)$$

with $R' = 2\rho'_+/ \ell$, the identifications (3.21) translate to

$$(u', \phi') \sim (u' + 2\pi i/c, \phi' - 2\pi d/c) \sim (u', \phi' + 2\pi), \quad (3.25)$$

while the thermal circle is independent of (c, d) ,

$$(u'(s), \phi'(s)) \sim (u'_0 + 2\pi i s, \phi'_0). \quad (3.26)$$

Comparing (3.25) with (3.14), it is apparent that the extremal limit of the solution labelled by (c, d) is a $\mathbb{Z}/c\mathbb{Z}$ orbifold of the solution (3.12), with R replaced by R' , by a translation

$$\gamma_c : (u, \phi) \mapsto (u + 2\pi i/c, \phi - 2\pi d/c). \quad (3.27)$$

Near $r = 1$, the Euclidean geometry looks like $(\mathbb{R}^2 \times S_1)/\mathbb{Z}_c$, where γ_c acts by a $2\pi/c$ rotation around the origin of the plane times a translation of angle $2\pi d/c$ along the circle S_1 parametrized by ϕ . Since (c, d) are relatively prime, this action has no fixed point and the quotient is smooth, as must be the case as the original family of solutions was smooth.

We now observe that the family of distinct extremal solutions given by (3.24) with periodicities (3.25), all have the same asymptotics, namely $AdS_2 \times S^1$, provided $1/R' = c/R + id$ is kept fixed while varying (R, c, d) . Indeed, in coordinates $r' = cr, u' = u/c, \phi' = \phi + i(d/c)u$, the metric takes the form

$$ds_3'^2 = \frac{\ell^2}{4} \left[- \left(r^2 - \frac{1}{c^2} \right) du^2 + \frac{dr^2}{r^2 - \frac{1}{c^2}} + R'^2 \left(d\phi + \left(r - \frac{1}{c} \right) \frac{du}{R'} - i \frac{d}{c} du \right)^2 \right], \quad (3.28)$$

with the same coordinate periodicities as in (3.14). It is easy to check that the fall-off conditions (2.4) with $v = \ell^2/4$ are indeed satisfied for any (c, d) coprime, $c \geq 1$. As mentioned in the introduction, in contrast to the AdS_3 Farey tail, the mass (or angular momentum) of the BTZ black hole is fixed, while the complex structure varies as a function of (c, d) . This is consistent with the microcanonical ensemble required for the quantum entropy function.

In the presence of fermions and if the original theory was supersymmetric, the extremal BTZ solutions admit Killing spinors [25]. These supercurrents depend only on the r coordinate in the solution (3.32), and therefore are not affected by the orbifold. Thus, the family of extremal solutions that we constructed preserve the same amount of supersymmetry.

3.4 The quantum entropy function for BTZ black holes

We can now apply the quantum entropy function formalism of § 2 to the extremal BTZ black hole. Since ϕ is a compact direction, the three-dimensional Einstein action may be reduced to two dimensions using the Kaluza-Klein ansatz

$$ds_3^2 = ds_2^2 + \ell^2 e^{-2\psi} (d\phi + \mathcal{A})^2, \quad (3.29)$$

leading to the two-dimensional action [26]

$$S = \int d^2x \sqrt{-g} \left[e^{-\psi} \left(R + \frac{2}{\ell^2} \right) - \frac{\ell^2}{4} e^{-3\psi} F^2 \right] + \dots, \quad (3.30)$$

where the ellipses denote contribution from the extra fields in three dimensions, and from Kaluza-Klein modes.

In particular, the BTZ metric (3.12) provides a classical solution to (3.30) with

$$ds_2^2 = \frac{\ell^2}{4} \left[-(r^2 - 1) du^2 + \frac{dr^2}{r^2 - 1} \right], \quad e^{-2\psi} = \frac{R^2}{4}, \quad \mathcal{A} = \frac{1}{R} (r - 1) du. \quad (3.31)$$

The two-dimensional metric ds_2^2 is just the two-dimensional AdS_2 black hole (2.1), with constant electric flux $e = 1/R$. Similarly, the family of solutions (3.28) reduces to solutions to (3.30) with the same asymptotics and charge as (3.31). This family of solutions will therefore also contribute to the quantum entropy function.

It is easiest to compute this contribution by regarding the (c, d) solution as a freely acting orbifold of the solution (3.31):

$$ds_2^2 = \frac{\ell^2}{4} \left[-(r^2 - 1) du^2 + \frac{dr^2}{r^2 - 1} \right], \quad e^{-2\psi} = \frac{R^2}{4}, \quad \mathcal{A} = \left(\frac{1}{R} (r - 1) + i d \right) du, \quad (3.32)$$

with $u \sim u + 2\pi i/c$. Since the solutions are locally isometric to (3.31), the Lagrangian density in coordinates (u, r) is constant and independent of (c, d) . In the classical limit (2.6), the (c, d) dependence appears in the periodicity of the u variable and the discrete Wilson line. The contribution of the bulk action (2.7) is

$$A_{\text{bulk}}^{(c,d)} = \frac{2\pi}{c} (r_0 - 1) v \mathcal{L} \quad (3.33)$$

where $v = \ell^2/4$, and the value of the Wilson line is

$$iq \oint_{u_E} \mathcal{A} = 2\pi i q d/c. \quad (3.34)$$

By putting a cut-off in the radial direction at $r = r_0$ in (3.32) (equivalently $r = r_0/c$ in (3.28)) and discarding the linearly divergent part, one finds that the solution labelled by (c, d) contributes in the classical limit to the quantum entropy function (2.5) as

$$\Omega^{(c,d)}(q) = \exp \left(\frac{\mathcal{S}_0}{c} + 2\pi i q \frac{d}{c} \right), \quad (3.35)$$

where \mathcal{S}_0 is the contribution for $(c, d) = (1, 0)$, i.e. the Wald entropy. Thus, the family of solutions with $c > 1$ leads to a series of exponentially suppressed corrections of the form (3.35).

Our conclusion agrees in spirit with the proposal put forward in [4, 5], but yields a more precise identification of the orbifold action in the case of BTZ black holes. In particular, the geometry associated to the subleading saddle points appears to be smooth, and the orbifold acts trivially on the compact manifold \tilde{M} . In the next section, we discuss the case of dyons in $\mathcal{N} = 4$ string backgrounds in more detail, where the compact manifold $M = S_1 \times S_1 \times S_2 \times K3$ allows for more general choices of the orbifold action.

4 The dyon partition function in $\mathcal{N} = 4$ theories

In this section, we study the exponentially subleading contributions to the degeneracies of dyons in $\mathcal{N} = 4$ string vacua, and find agreement with the general structure found in section 3. We also develop a general “Farey tale” expansion for the partition function of $\mathcal{N} = 4$ dyons at fixed value of the magnetic charge $P^2/2$, and contrast it with the usual “Farey tail” series governing the AdS_3 partition function.

4.1 The degeneracy formula

We first summarize some well-known facts about dyon degeneracies in $\mathcal{N} = 4$ string backgrounds, referring e.g. to [27] for more details. While our construction can be easily extended to other $\mathcal{N} = 4$ backgrounds, we focus for simplicity on the heterotic string compactified on a six-dimensional torus T^6 , or equivalently type II string on $K3 \times T^2$. The resulting four-dimensional theory is invariant under S and T-duality,

$$G(\mathbb{Z}) \equiv \text{SL}(2, \mathbb{Z}) \times O(22, 6; \mathbb{Z}) . \tag{4.1}$$

Dyons are labelled by their electric and magnetic charges $(Q^i, P^i), i = 1, \dots, 28$, transforming linearly as a **(2, 28)** representation of $G(\mathbb{Z})$. Both Q^i and P^i take values in an even self-dual lattice Λ of signature (22,6), the Narain lattice of the heterotic torus. The automorphism group of Λ defines the discrete subgroup $O(22, 6; \mathbb{Z}) \subset O(22, 6, \mathbb{R})$. The orbits of (Q^i, P^i) under $O(22, 6; \mathbb{Z})$ are labelled by the quadratic combinations $Q^2/2, P^2/2$, and $P \cdot Q$, invariant under the continuous T-duality, and the discrete invariant $I = \text{gcd}(Q^i P^j - Q^j P^i) \in \mathbb{Z}^+$, which is also invariant under S-duality. All dyons in the same orbit carry the same indexed degeneracy $\Omega(Q^2/2, P \cdot Q, P^2/2, I)$.³ Here we restrict to the simplest case $I = 1$, referring to [5, 16, 17] for generalizations.

The indexed degeneracies $\Omega(Q^2/2, P \cdot Q, P^2/2, I = 1)$ can be packaged into a partition function $\mathcal{Z}(\rho, v, \sigma)$, a function of three complex variables (ρ, v, σ) acting as chemical potentials for the T-duality invariants $(Q^2/2, P \cdot Q, P^2/2)$, respectively. As first conjectured in [10], \mathcal{Z} is a Siegel modular form of weight $k = -10$, i.e. it satisfies

$$\mathcal{Z}[(A\tau + B)(C\tau + D)^{-1}] = [\det(C\tau + D)]^k \mathcal{Z}(\tau) \tag{4.2}$$

for $k = 10$, where $\tau = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$ parametrizes Siegel’s upper half-plane

$$\text{Im}\rho > 0, \quad \text{Im}\sigma > 0, \quad (\text{Im}\rho)(\text{Im}\sigma) > (\text{Im}v)^2, \tag{4.3}$$

and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is any element of $\text{Sp}(2, \mathbb{Z})$, i.e. any integer valued matrix such that $gJg^t = J$ where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$:

$$AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = \mathbf{1} . \tag{4.4}$$

³For brevity we omit the dependence of Ω on the values of the moduli at spatial infinity, and correspondingly the ambiguity in the choice of integration contour in (4.9). The resulting ambiguities in Ω scale like $\exp(Q)$ and are still much smaller than the exponentially suppressed corrections of interest for this paper.

More specifically, for the heterotic string compactified on T^6 , \mathcal{Z} is the inverse of Igusa's cusp form Φ_{10} , which is the unique cusp form of weight $-k = 10$ under $\text{Sp}(2, \mathbb{Z})$:

$$\mathcal{Z} = \frac{1}{\Phi_{10}}. \quad (4.5)$$

Alternatively, Φ_{10} can be obtained as the square of the product of all even genus 2 theta functions, or as the additive lift of the index 1, weight 10 Jacobi form $\eta^{18}(\rho) \theta_1^2(\rho, v)$, or as the multiplicative lift of the elliptic genus of K3. The latter characterization means that [28]

$$\mathcal{Z} = \frac{\exp\left(\sum_{m=1}^{\infty} e^{2\pi i m \sigma} V_m \cdot \chi_{K3}\right)}{\eta^{18}(\rho) \theta_1^2(\rho, v)} \quad (4.6)$$

where

$$\chi_{K3}(\rho, v) = 24 \left(\frac{\theta_3(\rho, v)}{\theta_3(\rho, 0)} \right)^2 - 2 \frac{[\theta_4^4(\rho, 0) - \theta_2^4(\rho, 0)] \theta_1^2(\rho, v)}{\eta^6(\rho)} \quad (4.7)$$

and V_m are Hecke operators, acting on the Fourier coefficients $c(N, l)$ of a Jacobi form ϕ of weight k via [29]

$$V_m \cdot \phi(\rho, v) = \sum_{N, l} \left(\sum_{d|(N, l, m)} d^{k-1} c(Nl/d^2, l/d) \right) e^{2\pi i(N\rho + lv)}. \quad (4.8)$$

Given the dyon partition function \mathcal{Z} , the indexed degeneracies can be found from

$$\Omega(P, Q) = (-1)^{P \cdot Q + 1} \int_{\mathcal{C}} d\rho dv d\sigma e^{-i\pi(Q^2\rho + 2P \cdot Qv + P^2\sigma)} \mathcal{Z}(\rho, v, \sigma) \quad (4.9)$$

where (for an appropriate choice of moduli at spatial infinity) the contour \mathcal{C} is given by

$$0 < \text{Re}(\rho) \leq 1, \quad 0 < \text{Re}(v) \leq 1, \quad 0 < \text{Re}(\sigma) \leq 1, \quad (4.10)$$

while $\text{Im}(\rho), \text{Im}(v), \text{Im}(\sigma)$ are fixed to some large positive value. In this framework, S-duality is realized as an $\text{SL}(2, \mathbb{Z})$ subgroup of $\text{Sp}(2, \mathbb{Z})$, under which (ρ, v, σ) transforms as a three-vector dual to $(P^2/2, P \cdot Q, Q^2/2)$:

$$g_S = \begin{pmatrix} a & -b & 0 & 0 \\ -c & d & 0 & 0 \\ 0 & 0 & d & c \\ 0 & 0 & b & a \end{pmatrix}, \quad \begin{pmatrix} \rho' \\ v' \\ \sigma' \end{pmatrix} = \begin{pmatrix} a^2 & -2ab & b^2 \\ -ac & ad + bc & -bd \\ c^2 & -2cd & d^2 \end{pmatrix} \begin{pmatrix} \rho \\ v \\ \sigma \end{pmatrix}. \quad (4.11)$$

Invariance of \mathcal{Z} under $\text{SL}(2, \mathbb{Z}) \subset \text{Sp}(2, \mathbb{Z})$ ensures that the right-hand side of (4.9) is invariant under S-duality. The reason for covariance under the full Siegel modular group is less clear, except in the context of string network constructions [12, 30].

Rather than extracting the Fourier coefficients of $\mathcal{Z}(\rho, v, \sigma)$ with respect to its three arguments, it is useful to consider the partition function $\mathcal{Z}_m(\rho, v)$ for black holes at fixed values of $m = P^2/2$, but arbitrary values of $Q^2/2$ and $P \cdot Q$:

$$\mathcal{Z}_m(\rho, v) = \int_{0+i\text{Im}(\sigma)}^{1+i\text{Im}(\sigma)} d\sigma \mathcal{Z}(\rho, v, \sigma) e^{-2\pi i m \sigma}, \quad (4.12)$$

where again $\text{Im}(\sigma)$ is kept fixed and large. The modular property (4.2) for the subgroup $\text{SL}(2, \mathbb{R})_\rho \times H_3$ of $\text{Sp}(2, \mathbb{Z})$ of matrices of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_\rho = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : (\rho', v', \sigma') = \left(\frac{a\rho + b}{c\rho + d}, \frac{v}{c\rho + d}, \sigma - \frac{cv^2}{c\rho + d} \right) \quad (4.13)$$

and

$$\tilde{T}_{\lambda, \mu, \kappa} = \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} \rho' \\ v' \\ \sigma' \end{pmatrix} = \begin{pmatrix} \rho \\ v + \mu + \lambda\rho \\ \sigma + \kappa + 2\lambda v + \lambda\mu + \lambda^2\rho \end{pmatrix}, \quad (4.14)$$

implies that $\mathcal{Z}_m(\rho, v)$ is a Jacobi form of weight $k = -10$ and index m , i.e. it satisfies⁴

$$\mathcal{Z}_m \left(\frac{a\rho + b}{c\rho + d}, \frac{v}{c\rho + d} \right) = (c\rho + d)^k e^{2\pi i \frac{mcv^2}{c\rho + d}} \mathcal{Z}_m(\rho, v), \quad (4.15)$$

$$\mathcal{Z}_m(\rho, v + \lambda\rho + \mu) = e^{-2\pi i m[\lambda(\mu + 2v) + \lambda^2\rho]} \mathcal{Z}_m(\rho, v). \quad (4.16)$$

$\mathcal{Z}_m(\rho, v)$ is however not a holomorphic function of v , since it has a second order pole at the theta divisor $v \in \mathbb{Z} + \rho\mathbb{Z}$. This pole cancels in the product

$$\mathcal{Z}_m^{5D}(\rho, v) = \eta^{18}(\rho) \theta_1^2(\rho, v) \mathcal{Z}_m(\rho, v), \quad (4.17)$$

which is a holomorphic Jacobi form of weight 0 and index $m+1$. Physically, $\mathcal{Z}_m^{5D}(\rho, v)$ is the elliptic genus of the D1-D5 superconformal field theory, counting 5D black hole microstates with $Q_1 Q_5 = m$, momentum n and angular momentum l ,

$$\mathcal{Z}_m^{5D}(\rho, v) = \sum_{n, l} \Omega^{5D}(Q_1 Q_5, n, l) e^{2\pi i(n\rho + lv)}. \quad (4.18)$$

The equation (4.17) can be used to systematically evaluate the asymptotic expansion of the 5D black hole degeneracy [31, 32]. Our interest will be on the meromorphic partition function $\mathcal{Z}_m(\rho, v)$, which should correspond to the elliptic genus of the SCFT dual to $\mathcal{N} = 4$ dyons.

4.2 Mapping the poles

Our aim will be to evaluate the contour integrals (4.9) and (4.12) by use of Cauchy's residue formula. In this subsection, we describe the pole structure of the partition function \mathcal{Z} , and find an explicit $\text{Sp}(2, \mathbb{Z})$ transformation which maps any of them to the standard diagonal divisor $v = 0$.

⁴These relations may fail if one crosses poles in the σ plane when deforming the contour back to its original location. This does not happen provided the imaginary part $\text{Im}\sigma_*$ for all poles is bounded from above, and the contour in (4.12) is chosen at a sufficiently large value of $\text{Im}\sigma$.

The partition function \mathcal{Z} is well known to have a second order pole⁵ at the diagonal divisor $v = 0$, where it behaves as

$$\mathcal{Z}(\rho, v, \sigma) = \frac{1}{v^2 g(\rho, \sigma)} + \mathcal{O}(v^0), \tag{4.19}$$

where

$$g(\rho, \sigma) = \eta^{24}(\rho) \eta^{24}(\sigma) \tag{4.20}$$

is a Hilbert modular form⁶ of weight $2 - k$. By $\text{Sp}(2, \mathbb{Z})$ invariance, \mathcal{Z} must have a second order pole at all images of the diagonal divisor, i.e. at the quadratic divisors

$$D(m^i, j, n_i; \Omega) \equiv m^2 - m^1 \rho + n_1 \sigma + n_2(\rho \sigma - v^2) + jv = 0, \tag{4.21}$$

where j is any odd integer and the 5 integers $M = (m^1, m^2, j, n_1, n_2)$ are constrained to satisfy

$$\Delta(M) \equiv j^2 + 4(m^1 n_1 + m^2 n_2) = 1. \tag{4.22}$$

The diagonal divisor $v = 0$ corresponds to $M = (0, 0, 1, 0, 0)$, with $\Delta(M) = 1$. The union of all quadratic divisors (4.21) with $\Delta(M) = 1$ defines the first Humbert surface [34]. The invariance of the constraint (4.22) can be made manifest by fitting M into an anti-symmetric anti-traceless bilinear form in \mathbb{C}^4 ,

$$M = \begin{pmatrix} 0 & -m^2 & \frac{j}{2} & n_1 \\ m^2 & 0 & m^1 & -\frac{j}{2} \\ -\frac{j}{2} & -m^1 & 0 & -n_2 \\ -n_1 & \frac{j}{2} & n_2 & 0 \end{pmatrix}, \tag{4.23}$$

such that M transforms as $M' = \Omega M \Omega^t$ and $\Delta(M) = 4\text{Pf}(M)$ is manifestly invariant. This realizes the local isomorphism $\text{Sp}(2) = \text{SO}(2, 3)$. Moreover, one may check that under a simultaneous $\text{Sp}(2, \mathbb{Z})$ action on M and Ω , (4.21) transforms with weight -1 ,

$$D(M'; \Omega') = [\det(C\Omega + D)]^{-1} D(M, \Omega). \tag{4.24}$$

It will be important to determine the residue of \mathcal{Z} on the general quadratic divisor (4.21). For this purpose, it suffices to find a $\text{Sp}(2, \mathbb{Z})$ transformation which maps $M_1 = (0, 0, 1, 0, 0)$ to an arbitrary $M = (m^1, m^2, j, n_1, n_2)$ satisfying

$$m^1 n_1 + m^2 n_2 = \frac{1 - j^2}{4}. \tag{4.25}$$

Moreover, we shall insist that the choice of this transformation is covariant with respect to $\text{SL}(2, \mathbb{Z})_\rho$. We shall restrict our attention to $(n_1, n_2) \neq (0, 0)$. It is then useful to choose coprime integers (k_1, k_2) such that

$$k_2 n_1 - k_1 n_2 = r, \tag{4.26}$$

⁵Our construction straightforwardly generalizes to Siegel modular forms of arbitrary weight k , with a pole of arbitrary order at $v = 0$, or to modular forms invariant under a finite index subgroup of the Siegel modular group.

⁶For our purposes, a Hilbert modular form of weight w is a function of ρ, σ which is a modular form of weight w in each argument, and invariant under the exchange $\rho \leftrightarrow \sigma$, see e.g. [33].

where r is the greatest common divisor of $(n_1, n_2) = r(n'_1, n'_2)$. When (4.25) is obeyed, r must divide $(1 - j^2)/4$. The solutions to (4.25) can then be parametrized as

$$m^1 = -\frac{j^2 - 1}{4r}k_2 + \alpha n'_2, \quad m^2 = \frac{j^2 - 1}{4r}k_1 - \alpha n'_1, \quad (4.27)$$

where both $\alpha n'_1$ and $\alpha n'_2$ must be integer. Since (n'_1, n'_2) are coprime, this amounts to requiring that α is integer. Note that (k_1, k_2) are defined up to the addition of an integer multiple of (n'_1, n'_2) : this can be reabsorbed into a shift of α by an integer multiple of $(j^2 - 1)/4r$, which is integer. We further define

$$\delta \equiv \alpha \pmod{r}. \quad (4.28)$$

Since $r|(j^2 - 1)/4$, we may further decompose $r = r_1 r_2$ into a product of relatively prime factors, where r_1 divides $(j + 1)/2$ and r_2 divides $(j - 1)/2$:

$$j + 1 = 2r_1 j_2, \quad j - 1 = 2r_2 j_1, \quad r_1 j_2 - r_2 j_1 = 1. \quad (4.29)$$

The most general solution is given by

$$j_1 = s_1 + r_1 L, \quad j_2 = s_2 + r_2 L, \quad j = 2rL + j_0 \quad (4.30)$$

where s_1, s_2 are fixed integers with $r_1 s_2 - r_2 s_1 = 1$, L is an arbitrary integer, and $j_0 \equiv r_1 s_2 + r_2 s_1$.

Having defined these number theoretic quantities, it is now easy to check that

$$h = \begin{pmatrix} 1 & (\alpha - \delta)/r \\ 0 & 1 \end{pmatrix}_\sigma \cdot \begin{pmatrix} -k_1 & -n_1/r \\ k_2 & n_2/r \end{pmatrix}_\rho \cdot \begin{pmatrix} j_2 & 0 & 0 & j_1 \\ \delta j_2 & j_2 & j_1 & \delta j_1 \\ r_2 m_1 & r_2 & r_1 & -r_1 m_1 \\ r_2 & 0 & 0 & r_1 \end{pmatrix} \quad (4.31)$$

is an element of $\text{Sp}(2, \mathbb{Z})$ mapping $M_1 = (0, 0, 1, 0, 0)$ into $M = (m^1, m^2, j, n_1, n_2)$. Clearly, h is ambiguous modulo right multiplication by an element in the stabilizer of M_1 , i.e. in the Hilbert modular group $\text{SL}(2, \mathbb{Z})_\rho \times \text{SL}(2, \mathbb{Z})_\sigma \ltimes (\rho \leftrightarrow \sigma)$. As we discuss shortly, the choice (4.31) has the advantage of being covariant with respect to $\text{SL}(2, \mathbb{Z})_\rho$.

Denoting by $\tau' = (\rho', v', \sigma')$ the image⁷ of τ under h^{-1} , the quadratic divisor (4.21) is therefore mapped to $D(M_1, \tau') = 0$, i.e. $v' = 0$. For later use, we record the Jacobian from (ρ, v, σ) to (ρ', v', σ') : it is given by $\partial\tau/\partial\tau' = [\det(C\tau + D)]^3$ where $\det(C\tau + D)$ equals

$$\frac{1}{r_1^2} \det \begin{pmatrix} \frac{j+1}{2}(k_2\rho + k_1) + \delta(n_2\rho + n_1) + rv & \alpha + \frac{j-1}{2}\delta - v \left(\frac{j+1}{2}k_2 + \delta n_2 \right) - r\sigma \\ n_2\rho + n_1 & -n_2v + \frac{j+1}{2} \end{pmatrix}. \quad (4.32)$$

⁷For comparison with other discussions in the literature, our change of variable reduces for $M = (0, 0, 1, 0, 1)$, $k_1 = -1, k_2 = \alpha = \delta = 0$ to

$$\rho' = \frac{\rho}{(v-1)^2 - \rho\sigma}, \quad v' = \frac{v(1-v) + \rho\sigma}{(v-1)^2 - \rho\sigma}, \quad \sigma' = \frac{\sigma}{(v-1)^2 - \rho\sigma}.$$

The change of variable for general M is too cumbersome to be displayed, but follows immediately from $\tau' = (A\tau + B)(C\tau + D)^{-1}$ with $h^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ the inverse of (4.31).

Having found a suitable $\text{Sp}(2, \mathbb{Z})$ transformation mapping the general divisor (4.21) back to the diagonal divisor $v = 0$, it is now straightforward to extract the residue of \mathcal{Z} on (4.21) using (4.19) in the primed coordinates. For example, expanding around $\sigma = \sigma_*$ where σ_* is the location of the pole in the σ plane,

$$\sigma_* = \frac{m^1 \rho - m^2 + n_2 v^2 - j v}{n_2 \rho + n_1}, \quad (4.33)$$

we have

$$\begin{aligned} \rho' &= -\frac{1}{\rho_0} (1 - r_2^2 (\sigma - \sigma_*) / \rho_0 + \dots), \\ v' &= -\frac{r}{\rho_0} (\sigma - \sigma_*) (1 - r_2^2 (\sigma - \sigma_*) / \rho_0 + \dots), \\ \sigma' &= \sigma_0 + r_1^2 (\sigma - \sigma_*) + \dots, \\ \det(C\tau + D) &= -\rho_0 (n_2' \rho + n_1') (1 + r_2^2 (\sigma - \sigma_*) / \rho_0 + \dots), \end{aligned} \quad (4.34)$$

where (ρ_0, σ_0) are the values of $(-1/\rho', \sigma')$ at $\sigma = \sigma_*$, namely

$$\begin{aligned} \rho_0 &= \frac{j+1}{2r_1^2} \frac{k_2 \rho + k_1}{n_2' \rho + n_1'} + \frac{r_2}{r_1} \frac{v}{n_2' \rho + n_1'} + \frac{r_2}{r_1} \delta, \\ \sigma_0 &= -\frac{j-1}{2r_2^2} \frac{k_2 \rho + k_1}{n_2' \rho + n_1'} - \frac{r_1}{r_2} \frac{v}{n_2' \rho + n_1'} + \frac{r_1}{r_2} \delta. \end{aligned} \quad (4.35)$$

Using the modular properties of $\mathcal{Z}(\rho, v, \sigma)$ and $g(\rho, \sigma)$, we conclude that on the quadratic divisor (4.21),

$$\begin{aligned} \mathcal{Z}(\rho, v, \sigma) &\sim \frac{(\rho')^{2-k} [\det(C\tau + D)]^{-k}}{(v')^2 g(-1/\rho', \sigma')} \\ &= \frac{(n_2' \rho + n_1')^{-k}}{r^2 (\sigma - \sigma_*)^2 g(\rho_0, \sigma_0)} \left[1 - \frac{(r_2^2 \partial_{\rho_0} + r_1^2 \partial_{\sigma_0}) g(\rho_0, \sigma_0)}{g(\rho_0, \sigma_0)} (\sigma - \sigma_*) + \dots \right]. \end{aligned} \quad (4.36)$$

It is important to note that the poles (4.33) are bounded from above in the σ plane: indeed, expressing m^1 in terms of m^2, j, n_1, n_2 using (4.22) and extremizing with respect to j , one obtains

$$\text{Im} \sigma_* \leq \frac{\text{Im} \rho}{4|n_2 \rho + n_1|^2} + \frac{(\text{Im} v)^2}{\text{Im} \rho}, \quad (4.37)$$

where the upper bound would be reached at $j/2 = n_2 \text{Re}(v) - (n_1 + n_2 \text{Re} \rho) \text{Im} v / \text{Im} \rho$. A further extremization with respect to n_1, n_2 leads to

$$\text{Im} \sigma_* \leq \frac{1}{4} \max(\text{Im} \rho, 1/\text{Im} \rho) + \frac{(\text{Im} v)^2}{\text{Im} \rho}, \quad (4.38)$$

which ensures that the Fourier coefficient (4.12) is indeed a Jacobi form, as discussed in footnote 4.

Alternatively, one may expand in the v plane around either of the two roots of (4.21)

$$v_{\pm} = \frac{1}{2n_2} \left(j \pm \sqrt{(n_1 \sigma - m^1 \rho + m^2)^2 - 4jn_2} \right). \quad (4.39)$$

The asymptotic expansion is obtained by replacing

$$\sigma - \sigma_* = \frac{n_2}{n_2\rho + n_1} (\mp(v_+ - v_-)(v - v_\pm) + (v - v_\pm)^2 + \dots) \quad (4.40)$$

and $v = v_\pm + (v - v_\pm)$ in the expansions above.

It is important that our choice of $\text{Sp}(2, \mathbb{Z})$ transformation is covariant under $\text{SL}(2, \mathbb{Z})_\rho$: if (ρ, v) transform as in (4.13) and if M transforms as

$$\begin{pmatrix} n_1 & k_1 & m^2 \\ n_2 & k_2 & -m^1 \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} n_1 & k_1 \\ n_2 & k_2 \end{pmatrix}, \quad (j, \alpha, \delta) \mapsto (j, \alpha, \delta), \quad (4.41)$$

then ρ_0, σ_0 are invariant, while σ_* transforms in the same way as σ in (4.13). On the other hand, under the spectral flow (4.14), n_1, n_2, k_1, k_2 can be taken to be invariant, so that

$$\begin{aligned} \rho_0 &\mapsto \rho_0 - \frac{r_2}{r_1}(k_1\lambda - k_2\mu), & \sigma_0 &\mapsto \sigma_0 + \frac{r_1}{r_2}(k_1\lambda - k_2\mu), \\ \sigma_* &\mapsto \sigma_* + \lambda(\mu + \lambda\rho + 2v) + \kappa, \end{aligned} \quad (4.42)$$

while α shifts by an integer,

$$\alpha \mapsto \alpha + \frac{(j - \lambda n_1 + \mu n_2)[k_2(\mu n_2 - \lambda n_1) + r\lambda]}{n_2} + \kappa r. \quad (4.43)$$

In particular, the spectral flow $\tilde{T}_{0,0,\kappa}$ takes $\alpha \mapsto \alpha + \kappa r$, and leaves δ invariant.

4.3 Subleading contributions to the 4D degeneracies

We now evaluate the integral (4.9) by first performing the σ integral using Cauchy's residue formula, and evaluating the remaining integral over (ρ, v) in the saddle point approximation.⁸ This approximation becomes exact in the limit where $Q^2/2, P^2/2, P \cdot Q$ are scaled to infinity keeping their ratio fixed. Moreover, we assume that the quartic invariant $P^2Q^2 - (P \cdot Q)^2$ is positive, such that the entropy is dominated by a large dyonic black hole.

The poles that contribute to the σ integral must belong to the strip $0 \leq \sigma \leq 1$. Since any pole can be mapped into this strip by a spectral flow $\tilde{T}_{0,0,\kappa}$, which maps $(m^1, m^2) \mapsto (m^1 + \kappa n_2, m^2 - \kappa n_1)$, and since the residue is invariant under this action, we must restrict to poles with $0 \leq m^1 < n_2$, subject to the quadratic constraint (4.22). Using (4.36), we find

$$\begin{aligned} \Omega(P, Q) &= \Omega^{(0)}(P, Q) + (-1)^{P \cdot Q + 1} \int_0^1 d\rho \int_0^1 dv \sum_{(n_1, n_2, j, m^1, m^2), \Delta(M)=1} \\ &\frac{(n'_2\rho + n'_1)^{-k}}{r^2 g(\rho_0, \sigma_0)} \left(i\pi P^2 + \frac{(r_2^2 \partial_{\rho_0} + r_1^2 \partial_{\sigma_0})g(\rho_0, \sigma_0)}{g(\rho_0, \sigma_0)} \right) e^{-i\pi(Q^2\rho + 2P \cdot Qv + P^2\sigma_*)}, \end{aligned} \quad (4.44)$$

where $\Omega^{(0)}(P, Q)$ includes the contribution with poles with $(n_1, n_2) = (0, 0)$, possibly together with additional boundary contributions which remain after the contour has been

⁸The more standard approach where the integral over v is done by the residue theorem while the ones over (ρ, σ) are done in the saddle point approximation is discussed in appendix B.

deformed across all poles with $(n_1, n_2) \neq (0, 0)$. As mentioned below (4.12), the integrand is a Jacobi form $\mathcal{Z}_m(\rho, v)$ of index $m = P^2/2$ and weight $k = -10$, which we discuss in its own right in the next subsection. For now, we proceed with the integral over ρ and v .

To leading order in the charges, the integral (4.44) can be approximated by extremizing the exponent with respect to (ρ, v) , including the fluctuation determinant and evaluating the prefactor at the saddle point. The saddle point lies at

$$\rho_* = -\frac{n_1}{n_2} + i \frac{P^2}{2n_2 \sqrt{P^2 Q^2 - (P \cdot Q)^2}}, \quad v_* = \frac{j}{2n_2} + i \frac{P \cdot Q}{2n_2 \sqrt{P^2 Q^2 - (P \cdot Q)^2}}, \quad (4.45)$$

at which point the location of the pole (4.33) evaluates to

$$\sigma_* = \frac{m^1}{n_2} + i \frac{Q^2}{2n_2 \sqrt{P^2 Q^2 - (P \cdot Q)^2}}. \quad (4.46)$$

Note in particular that (ρ_*, v_*, σ_*) transforms as a triplet under S-duality (4.11). For the saddle point to lie in the integration domain, we require that $-n_2 < n_1 \leq 0$, $1 \leq j < 2n_2$. The value of the exponent at the saddle point is given by [4]

$$\mathcal{S}_* = \frac{\pi}{n_2} \sqrt{P^2 Q^2 - (P \cdot Q)^2} + \frac{i\pi}{n_2} (n_1 Q^2 - j(P \cdot Q) - m^1 P^2) \quad (4.47)$$

Thus, as already noted in [10], the poles with $n_2 > 1$ give exponentially suppressed contributions with respect to the one with $n_2 = 1$. Our interest is in further analyzing the contributions of all the subleading saddle points, extending the analysis in [4].

The fluctuation determinant around the saddle point (4.45) is given by

$$\det \begin{pmatrix} \partial_{\rho^2} \mathcal{S} & \partial_{\rho v} \mathcal{S} \\ \partial_{\rho v} \mathcal{S} & \partial_{v^2} \mathcal{S} \end{pmatrix} = \left(\frac{4\pi n_2 [P^2 Q^2 - (P \cdot Q)^2]}{P^2} \right)^2. \quad (4.48)$$

Moreover, the arguments (ρ_0, σ_0) of the prefactor in (4.44) reduce to

$$\begin{aligned} \rho_0^* &= \frac{k_2(j+1)}{2r_1^2 n_2'} + \frac{1}{r_1 n_2' P^2} \left(-P \cdot Q + i \sqrt{P^2 Q^2 - (P \cdot Q)^2} \right), \\ \sigma_0^* &= -\frac{k_2(j-1)}{2r_2^2 n_2'} + \frac{1}{r_2 n_2' P^2} \left(P \cdot Q + i \sqrt{P^2 Q^2 - (P \cdot Q)^2} \right). \end{aligned} \quad (4.49)$$

Thus, the saddle point labelled by (n_1, n_2, j, m^1, m^2) with j odd and

$$\Delta(M) = 1, \quad 0 \leq m^1 < n_2, \quad -n_2 < n_1 \leq 0, \quad 1 \leq j < 2n_2, \quad (4.50)$$

contributes to the degeneracies of $\mathcal{N} = 4$ in the semi-classical limit as

$$\frac{(-1)^{P \cdot Q} (P^2)^{1-k}}{r^{2-k} n_2 [P^2 Q^2 - (P \cdot Q)^2]^{1-\frac{k}{2}} g(\rho_0^*, \sigma_0^*)} \left(i\pi P^2 + \frac{(r_2^2 \partial_{\rho_0^*} + r_1^2 \partial_{\sigma_0^*}) g(\rho_0^*, \sigma_0^*)}{g(\rho_0^*, \sigma_0^*)} \right) e^{\mathcal{S}_*} \quad (4.51)$$

When $n_2 = |j| = 1$ and therefore $n_1 = m^1 = m^2 = k_2 = \alpha = \delta = 0, r = -k_1 = 1$, the exponent \mathcal{S}_* in (4.47) reproduces the Bekenstein-Hawking entropy of the dyonic black hole.

The prefactor in (4.51) leads to logarithmically and power suppressed corrections to the entropy which are consistent with the contributions of the R^2 -type quantum corrections to the Wald entropy [11]. In fact, the values of (ρ_0, σ_0) in (4.49) are equal to the attractor values of the axio-dilaton $(S, -\bar{S})$, and the functions g_1, g_2 are precisely the ones which govern the R^2 corrections to the four-dimensional effective action [35, 36]. This agreement continues to hold off-shell [11], as outlined in appendix B.

Instead, contributions with $n_2 > 1$ are exponentially suppressed compared to those with $n_2 = 1$. They agree qualitatively with the semi-classical contributions of subleading AdS_2 saddle points (3.35) computed in section 3 (strictly speaking for n_1, n_2 coprime only). While (3.35) involved only one discrete quantum number d conjugate to a single charge q , (4.47) displays the contributions of three discrete quantum numbers $n_1, -j/2, -m^1$ conjugate to $Q^2/2, P \cdot Q, P^2/2$, respectively. In the duality frame of the $D1 - D5 - P - KKM$ system on $S^1 \times S^1 \times K3$ with one unit of KK monopole charge, the quadratic invariants $Q^2/2 = q, P \cdot Q = l$ become equal to the momenta along the two circles, and so n_1 and $j/2$ may be viewed as the timelike component of the Kaluza-Klein gauge fields $g_{\mu 5}$ and $g_{\mu 6}$ along the two circles. The third combination $m^1 P^2/2 = m^1 Q_1 Q_5$ for either $Q_1 = 1$ or $Q_5 = 1$ may be interpreted as a discrete Wilson line for the Ramond-Ramond one-form or five-form. When $n_1, (j \pm 1)/2, m_1$ and n_2 are not relatively prime, the orbifold described in section 3 is no longer freely acting, but nevertheless the saddle point action still retains the same form (4.47).

In principle, it should also be possible to interpret the prefactor in (4.51) as the effect of R^2 -type corrections around the subleading semi-classical geometry. For (n_1, n_2) coprime, and therefore $r_1 = r_2 = 1$, the values of the “axio-dilaton” (ρ_0, σ_0) appearing as the argument of the Hilbert modular function g are rescaled by a factor $1/n_2$ compared to the values at the leading saddle point. If the heterotic coupling is weak in the attractor region, one may approximate $\ln g(\rho_0^*, \sigma_0^*) \sim 2\pi i(\rho_0^* + \sigma_0^*)$, which is reduced by a factor $1/n_2$ from its value at $(n_1, n_2) = (0, 1)$. This appears to be consistent with our identification of the subleading saddle point as a $\mathbb{Z}/n_2\mathbb{Z}$ orbifold of the dominant solution. More generally, this is consistent with the fact that the effective volume of the two-torus $S_1 \times S_1$ is effectively reduced by a factor n_2 at the “very near horizon” $r = 1$. We do not know how to interpret the additional shift proportional to k_2 , which breaks the reality relation $\rho_0 = -\bar{\sigma}_0$. We also explain in appendix B that the off-shell agreement with the entropy function which was observed in [11] for the leading saddle point does not seem to extend to subleading saddle points.

4.4 Poincaré series representation

We now consider the partition function (4.12) at fixed $m = P^2/2$. As explained below (4.12), this is a meromorphic Jacobi form of weight $k = -10$ and index m , which should be identified as the elliptic genus of the SCFT dual to 4D dyons. In the same way as in section 4.3, the integral over σ can be evaluated by Cauchy’s residue formula, leading

to

$$\mathcal{Z}_m(\rho, v) = \sum \frac{2\pi i (n'_2 \rho + n'_1)^{-k}}{r^2 g(\rho_0, \sigma_0)} \left(2\pi i m + \frac{(r_2^2 \partial_{\rho_0} + r_1^2 \partial_{\sigma_0}) g(\rho_0, \sigma_0)}{g(\rho_0, \sigma_0)} \right) \times \exp \left[-2\pi i m \left(\frac{m^1 \rho - m^2}{n_2 \rho + n_1} + \frac{n_2 v^2 - jv}{n_2 \rho + n_1} \right) \right] \quad (4.52)$$

where the sum runs over all integers $j, n_1, n_2, k_1, k_2, \delta, r_1, r_2$ (the condition $0 \leq m^1 < m^2$ fixes α uniquely, but its precise value is irrelevant). Fourier expanding the prefactor,

$$\frac{2\pi i m g(\rho_0, \sigma_0) + (r_2^2 \partial_{\rho_0} + r_1^2 \partial_{\sigma_0}) g(\rho_0, \sigma_0)}{r^2 (g(\rho_0, \sigma_0))^2} = \sum_{N_1, N_2} c_{m, r_1, r_2}(N_1, N_2) e^{2\pi i (N_1 \rho_0 + N_2 \sigma_0)} \quad (4.53)$$

we obtain

$$\mathcal{Z}_m(\rho, v) = 2\pi i \sum_{j, n_1, n_2, k_1, k_2, r_1, r_2, N_1, N_2, \delta} (n'_2 \rho + n'_1)^{-k} c_{m, r_1, r_2}(N_1, N_2) e^{\mathcal{S}} \quad (4.54)$$

where the exponent may be written, up to an additive integer, as

$$\frac{\mathcal{S}}{2\pi i} = -\frac{m n'_2 v^2}{n'_2 \rho + n'_1} + \frac{lv}{n'_2 \rho + n'_1} + \frac{k_2 \rho + k_1}{n'_2 \rho + n'_1} \left(\tilde{N} + \frac{l^2}{4m} \right) + (r_2^2 N_1 + r_1^2 N_2 - m) \frac{\delta}{r}, \quad (4.55)$$

where we have defined

$$l = \frac{jm}{r} + r \left(\frac{N_1}{r_1^2} - \frac{N_2}{r_2^2} \right), \quad \tilde{N} = \frac{1}{2} \left(\frac{N_1}{r_1^2} + \frac{N_2}{r_2^2} - \frac{m}{2r^2} \right) - \frac{(N_1 r_2^2 - N_2 r_1^2)^2}{4mr^2} \quad (4.56)$$

The sum over δ ranging from 0 to $r - 1$ vanishes unless $r_2^2 N_1 + r_1^2 N_2 - m$ is divisible by r , in which case it produces an overall factor of r . To solve the congruence, let us choose integers t_1, t_2 such that

$$m = r_1^2 t_2 + r_2^2 t_1. \quad (4.57)$$

Since r_1 and r_2 are coprime, they must divide $N_1 - t_1$ and $N_2 - t_2$, respectively:

$$N_1 = t_1 + r_1 N'_1, \quad N_2 = t_2 + r_2 N'_2, \quad (4.58)$$

where N'_1, N'_2 are integers. Moreover, using (4.29) and (4.57), we can write

$$l = \mu + 2mL, \quad \mu \equiv r_2 t_1 \frac{j_0 + 1}{r_1} - r_1 t_2 \frac{1 - j_0}{r_2} + N'_1 r_2 - N'_2 r_1, \quad (4.59)$$

where μ is manifestly integer. Having defined μ in this way, one may further compute $N \equiv \tilde{N} + \frac{\mu^2}{4m}$,

$$N = \frac{1 + j_0}{2r_1} N'_1 + \frac{1 - j_0}{2r_2} N'_2 + \frac{(1 + j_0)^2}{4r_1^2} t_1 + \frac{(1 - j_0)^2}{4r_2^2} t_2, \quad (4.60)$$

which is also manifestly integer.

The sum over L produces a unary theta series (A.3) evaluated at $\rho' = (k_2\rho + k_1)/(n'_2\rho + n'_1)$, $v' = v/(n'_2\rho + n'_1)$. Therefore, identifying

$$a = k_2, \quad b = k_1, \quad c = n'_2, \quad d = n'_1, \tag{4.61}$$

we recognize the sum over poles (4.52) as a Poincaré series,

$$\phi_m = \phi_m^{(0)} + \frac{1}{2} \sum_{\mu=0}^{2m-1} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (c\rho + d)^{-k} e^{-\frac{2i\pi m c v^2}{c\rho + d}} h_\mu \left(\frac{a\rho + b}{c\rho + d} \right) \theta_{m,\mu} \left(\frac{a\rho + b}{c\rho + d}, \frac{v}{c\rho + d} \right), \tag{4.62}$$

where $\phi_m^{(0)}$ denotes the contributions of the poles with $(n_1, n_2) = 0$, and

$$h_\mu(\rho) = 2\pi i r \sum_{r_1, r_2, N'_1, N'_2} c_{m, r_1, r_2} (r_1 N'_1 + t_1, r_2 N'_2 + t_2) e^{2\pi i \left(N - \frac{\mu^2}{4m} \right) \rho} \tag{4.63}$$

where the sum runs over integers with a fixed value of $\mu \bmod 2m$. Finally, the original Siegel modular form \mathcal{Z} may be recovered by resumming the Fourier series,

$$\mathcal{Z} = \mathcal{Z}^{(0)} + \frac{1}{2} \sum_m \sum_{\mu=0}^{2m-1} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (c\rho + d)^{-k} e^{2\pi i m \left(\sigma - \frac{c v^2}{c\rho + d} \right)} h_\mu \left(\frac{a\rho + b}{c\rho + d} \right) \theta_{m,\mu} \left(\frac{a\rho + b}{c\rho + d}, \frac{v}{c\rho + d} \right) \tag{4.64}$$

To summarize, we have rewritten the sum over the five integers $M = (m^1, m^2, j, n_1, n_2)$ modulo the constraint $\Delta(M) = 1$ into a sum over cosets $\begin{pmatrix} k_2 & k_1 \\ n_2/r & n_1/r \end{pmatrix} \in \Gamma_\infty \backslash \Gamma$, spectral flow $l = \mu \bmod 2m$, and quantum numbers r_1, r_2, N'_1, N'_2 . The auxiliary integers s_1, s_2, t_1, t_2 are fixed in terms of m, r_1, r_2 by $r_1 s_2 - r_2 s_1 = 1$ and (4.57).

The Poincaré series (4.62) is our “Farey tale” expansion for the $\mathcal{N} = 4$ dyon partition function. It closely resembles the usual Farey tail expansion (A.5) for Jacobi modular forms. While the sum (4.62) is not restricted to states satisfying the “cosmic censorship” bound $\tilde{N} < 0$, this restriction may be enforced by hand at no cost, since non-polar terms (after properly regulating the sum) average to zero [9].

In detail however, the structure of (4.62) is considerably more intricate than (A.5). In particular, the Fourier coefficients of the vector-valued modular form $h_\mu(\rho)$ must take the special form (4.63), where the coefficients $c_{m, r_1, r_2}(N_1, N_2)$ originate from expanding (4.53). This structure ensures that the resulting sum (4.64) is (at least formally) a Siegel modular form of weight k , for any choice of a Hilbert modular form $g(\rho, \sigma)$ of weight $2 - k$. In fact, the series (4.64) is a Poincaré-type series for the Siegel modular group, where the sum runs over the coset $\Gamma_1 \backslash \text{Sp}(2, \mathbb{Z})$, where Γ_1 is the stabilizer of $M_1 = (0, 0, 1, 0, 0)$, i.e. the Hilbert modular group. Thus, it should provide a lift from Hilbert modular forms to Siegel modular forms with a pole on the diagonal divisor $v = 0$.

Unfortunately, in contrast to (A.5), our expansion (4.62) is formal, as we have not attempted to regulate the sum over poles. In particular, we have little control over the “degenerate contribution” $\mathcal{Z}^{(0)}$, which is the part that remains once the contour in the σ plane has been passed through all the poles with $\text{Im}\sigma_* > 0$. This can in principle be

determined from the “non-degenerate” contributions $(n_1, n_2) \neq 0$ by requiring that (4.64) is $\text{Sp}(2, \mathbb{Z})$ invariant. This degenerate contribution should reproduce the expected pole (4.19) at $v = 0$, together with its images at $v = m^1 \rho - m^2$. Thus, it is natural to expect that it is given by the sum

$$\sum_{(m^1, m^2) \in \mathbb{Z}^2} \frac{1}{(v - m^1 \rho + m^2)^2 g(\rho, \sigma - m^1(m^1 \rho - m^2))}, \quad (4.65)$$

while $\phi_m^{(0)}$ will be given by the Fourier coefficients of this sum with respect to σ . It is an interesting mathematical problem to turn our “Farey tale” into a precise mathematical statement, and see whether additional constraints must be imposed on $g(\rho, \sigma)$ to avoid possible modular anomalies, in the spirit of [9].

Assuming that the Farey tale expansion can be made rigorous, it gives an alternative representation of the Fourier-Jacobi coefficients of $1/\Phi_{10}$ in terms of the Fourier coefficients of the Dedekind modular form, as opposed to the standard Farey tail representation (A.5) where h_μ are directly related to the coefficients of the elliptic genus of $K3$ (4.7) via the action of the Hecke operators (4.8). The agreement between the two representations implies identities between these Fourier coefficients which would be interesting to spell out.

From the physics point of view, it should be possible to give a detailed macroscopic interpretation of (4.62) as a sum over AdS_2 geometries. We have offered an interpretation of the exponent \mathcal{S}_* at the saddle point, but clearly more work remains to interpret the prefactor especially when n_1, n_2 are not relatively prime. Moreover, it would also be desirable to improve our understanding of the degenerate contributions, as they play crucial role for consistency with wall-crossing [37]. Interestingly, this is tied with the fact that the decomposition (A.2) for meromorphic forms involves vector-valued “mock” modular forms rather than usual modular forms [38–40]. Hopefully, resolving these issues will shed some light on the microscopic description of $\mathcal{N} = 4$ dyons.

Acknowledgments

We are grateful to A. Castro, A. Dabholkar, S. Minwalla, E. Verlinde and A. Sen for discussions. The research of B.P. is supported in part by ANR (CNRS-USAR) contract no.05-BLAN-0079-01. The research of S.M. is supported in part by the European Commission Marie Curie Fellowship under the contract PIIF-GA-2008-220899. S.M. would like to thank TIFR, Mumbai for hospitality where part of this work was carried out.

A Review of the Farey tail expansion

For the reader’s convenience, we briefly summarize the Farey tail expansion of a weak holomorphic Jacobi form $\phi(\rho, v)$ of weight $k \leq 0$ and index m [9, 24]. Let $c(N, l)$ be the Fourier coefficients of ϕ ,

$$\phi(\rho, v) = \sum_{N \geq 0, l \in \mathbb{Z}} c(N, l) e^{2\pi i(N\rho + lv)}. \quad (A.1)$$

Using spectral flow invariance, ϕ can be decomposed as

$$\phi(\rho, v) = \sum_{\mu=0}^{2m-1} h_{\mu}(\rho) \theta_{m,\mu}(\rho, v), \tag{A.2}$$

where $\theta_{m,\mu}$ is an index m unary theta series,

$$\theta_{m,\mu}(\rho, v) = \sum_{l \in \mathbb{Z}, l \equiv \mu \pmod{2m}} e^{2\pi i \left(\frac{l^2}{4m} \rho + lv \right)}, \tag{A.3}$$

and $h_{\mu}(\rho)$ is a vector valued modular form

$$h_{\mu}(\rho) = \sum_{N \in \mathbb{Z}} c_{\mu}(4mN - \mu^2) e^{2\pi i \left(N - \frac{\mu^2}{4m} \right) \rho} \tag{A.4}$$

where $c_{\mu}(4mN - \mu^2) \equiv (-1)^{2ml} c(N, l)$ for $l \equiv \mu \pmod{2m}$. Note that the holomorphy of ϕ with respect to v is essential: if ϕ has poles in the v -plane, $h_{\mu}(\rho)$ are only “mock” modular forms, and (A.2) has to be supplemented an extra term [38].

The Farey tail expansion of ϕ can be obtained by replacing $h_{\mu}(\rho)$ in (A.2) by its Rademacher expansion. In this way one obtains the Poincaré series representation

$$\begin{aligned} \phi_m = \frac{1}{2} h_{\mu} \left(\frac{\mu^2}{4m} \right) \theta_{\mu}(\rho, v) + \frac{1}{2} \sum_{\mu=0}^{2m-1} \sum_{\gamma \in \Gamma_{\infty} \setminus \Gamma} (c\rho + d)^{-k} e^{-\frac{2i\pi m c v^2}{c\rho + d}} \\ \times h_{\mu}^{-} \left(\frac{a\rho + b}{c\rho + d} \right) \theta_{m,\mu} \left(\frac{a\rho + b}{c\rho + d}, \frac{v}{c\rho + d} \right) \end{aligned} \tag{A.5}$$

where h_{μ}^{-} is the regularized polar part,

$$h_{\mu}^{-}(\rho) = \sum_{N; 4mN - \mu^2 < 0} c_{\mu}(4mN - \mu^2) e^{2\pi i \left(N - \frac{\mu^2}{4m} \right) \rho} R \left(\frac{2\pi i |N - \frac{\mu^2}{4m}|}{c(c\rho + d)} \right) \tag{A.6}$$

and $R(x)$ is a regularizing factor, such that $R(x) \rightarrow 1$ exponentially fast at $x \rightarrow \infty$. The sum runs over $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $(c, d) = 1$ and (a, b) is any one of the solutions of $ad - bc = 1$. It may be regularized by restricting to $|c| \leq K, |d| \leq K$ and letting $K \rightarrow \infty$ at the end. Note that due to the regularization, the Poincaré sum for arbitrary choices of the polar coefficients c_{μ} and multiplier system may not be modular invariant. It is possible however to supplement it with a non-holomorphic term so as to restore modular invariance [9].

B Subleading contributions to the entropy function

In this appendix, we evaluate (4.9) by a more standard procedure [11, 15], which is to first perform the integral over v using Cauchy residue’s formula, and then treat the integral over ρ, σ by saddle point methods. The advantage is that, keeping only the term with $n_2 = 1$ in the first step, and identifying (ρ, σ) with the axio-dilaton $S = S_1 + iS_2$ according to via

$$\rho = \frac{i}{2S_2}, \quad \sigma = \frac{i(S_1^2 + S_2^2)}{2S_2}, \tag{B.1}$$

the integrand may be recognized as the exponential of the macroscopic entropy function, with contributions from R^2 -type corrections to the 4D low energy effective action [41].

This off-shell agreement between the microscopic partition function and the macroscopic geometry is quite remarkable. However, it does not seem to extend to exponentially suppressed corrections with $n_2 > 1$. Indeed, at the pole $v = v_+$ in (4.39), the exponent in (4.9) becomes

$$\mathcal{S} = \frac{\pi}{2S_2} |Q - \tau P|^2 - i\pi\nu P \cdot Q, \quad (\text{B.2})$$

with

$$\nu = \frac{j}{n_2} + \frac{iS_1}{S_2} - \frac{1}{n_2 S_2} \sqrt{S_2^2 - (n_2 - 2in_1 S_2)(n_2(S_1^2 + S_2^2) + 2im_1 S_2)}, \quad (\text{B.3})$$

while the arguments of the Hilbert modular form $g(\rho_0, \sigma_0)$ appearing in (4.36) are given by

$$\begin{aligned} \rho_0 &= \frac{(1+j)r_2 k_2}{2r_1 n_2} + \frac{ir_2^2}{n_2(n_2 - 2in_1 S_2)} \left(S_2 + \sqrt{S_2^2 - (n_2 - 2in_1 S_2)(n_2(S_1^2 + S_2^2) + 2im_1 S_2)} \right) \\ \sigma_0 &= \frac{(1-j)r_1 k_2}{2r_2 n_2} + \frac{ir_1^2}{n_2(n_2 - 2in_1 S_2)} \left(S_2 - \sqrt{S_2^2 - (n_2 - 2in_1 S_2)(n_2(S_1^2 + S_2^2) + 2im_1 S_2)} \right) \end{aligned} \quad (\text{B.4})$$

Thus, the integrand in the ρ, σ integral can be written as e^{-F} where

$$-F = \frac{\pi}{2S_2} |Q - \tau P|^2 - i\pi P \cdot Q (\nu - 1) - \log g(\rho_0, \sigma_0) + \dots \quad (\text{B.5})$$

where the ellipses stand for other contributions to the residue at $v = v_{\pm}$. For the leading order contribution with $(n_1, n_2) = (0, 1)$, (B.3) and (B.4) reduce to

$$\nu = 1, \quad \rho_0 = -S_1 + iS_2, \quad \sigma_0 = S_1 + iS_2, \quad (\text{B.6})$$

and F is recognized as entropy function of the four-dimensional black hole, including the effect of the R^2 -type corrections with dependence on the axio-dilaton S [11, 15]. For $n_2 > 1$, this interpretation of the (ρ, σ) integral seems to break down.

References

- [1] A. Sen, *Quantum entropy function from AdS_2/CFT_1 correspondence*, [arXiv:0809.3304](#) [[SPIRES](#)].
- [2] J.M. Maldacena and A. Strominger, *AdS_3 black holes and a stringy exclusion principle*, *JHEP* **12** (1998) 005 [[hep-th/9804085](#)] [[SPIRES](#)].
- [3] M. Spradlin and A. Strominger, *Vacuum states for AdS_2 black holes*, *JHEP* **11** (1999) 021 [[hep-th/9904143](#)] [[SPIRES](#)].
- [4] N. Banerjee, D.P. Jatkar and A. Sen, *Asymptotic expansion of the $N = 4$ dyon degeneracy*, *JHEP* **05** (2009) 121 [[arXiv:0810.3472](#)] [[SPIRES](#)].
- [5] A. Sen, *Arithmetic of quantum entropy function*, *JHEP* **08** (2009) 068 [[arXiv:0903.1477](#)] [[SPIRES](#)].
- [6] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379** (1996) 99 [[hep-th/9601029](#)] [[SPIRES](#)].

- [7] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, *JHEP* **12** (1997) 002 [[hep-th/9711053](#)] [[SPIRES](#)].
- [8] P. Kraus and F. Larsen, *Partition functions and elliptic genera from supergravity*, *JHEP* **01** (2007) 002 [[hep-th/0607138](#)] [[SPIRES](#)].
- [9] J. Manschot and G.W. Moore, *A modern Farey tail*, [arXiv:0712.0573](#) [[SPIRES](#)].
- [10] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *Counting dyons in $N = 4$ string theory*, *Nucl. Phys. B* **484** (1997) 543 [[hep-th/9607026](#)] [[SPIRES](#)].
- [11] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Asymptotic degeneracy of dyonic $N = 4$ string states and black hole entropy*, *JHEP* **12** (2004) 075 [[hep-th/0412287](#)] [[SPIRES](#)].
- [12] D. Gaiotto, *Re-counting dyons in $N = 4$ string theory*, [hep-th/0506249](#) [[SPIRES](#)].
- [13] D. Shih, A. Strominger and X. Yin, *Recounting dyons in $N = 4$ string theory*, *JHEP* **10** (2006) 087 [[hep-th/0505094](#)] [[SPIRES](#)].
- [14] D. Shih and X. Yin, *Exact black hole degeneracies and the topological string*, *JHEP* **04** (2006) 034 [[hep-th/0508174](#)] [[SPIRES](#)].
- [15] J.R. David and A. Sen, *CHL dyons and statistical entropy function from D1-D5 system*, *JHEP* **11** (2006) 072 [[hep-th/0605210](#)] [[SPIRES](#)].
- [16] S. Banerjee, A. Sen and Y.K. Srivastava, *Partition functions of torsion > 1 dyons in heterotic string theory on T^6* , *JHEP* **05** (2008) 098 [[arXiv:0802.1556](#)] [[SPIRES](#)].
- [17] A. Dabholkar, J. Gomes and S. Murthy, *Counting all dyons in $N = 4$ string theory*, [arXiv:0803.2692](#) [[SPIRES](#)].
- [18] A. Castro, D. Grumiller, F. Larsen and R. McNees, *Holographic description of AdS_2 black holes*, *JHEP* **11** (2008) 052 [[arXiv:0809.4264](#)] [[SPIRES](#)].
- [19] A. Sen, *Entropy function and AdS_2/CFT_1 correspondence*, *JHEP* **11** (2008) 075 [[arXiv:0805.0095](#)] [[SPIRES](#)].
- [20] M. Bañados, C. Teitelboim and J. Zanelli, *The black hole in three-dimensional space-time*, *Phys. Rev. Lett.* **69** (1992) 1849 [[hep-th/9204099](#)] [[SPIRES](#)].
- [21] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the $(2 + 1)$ black hole*, *Phys. Rev. D* **48** (1993) 1506 [[gr-qc/9302012](#)] [[SPIRES](#)].
- [22] S. Carlip, *The $(2 + 1)$ -dimensional black hole*, *Class. Quant. Grav.* **12** (1995) 2853 [[gr-qc/9506079](#)] [[SPIRES](#)].
- [23] R.K. Gupta and A. Sen, *AdS_3/CFT_2 to AdS_2/CFT_1* , *JHEP* **04** (2009) 034 [[arXiv:0806.0053](#)] [[SPIRES](#)].
- [24] R. Dijkgraaf, J.M. Maldacena, G.W. Moore and E.P. Verlinde, *A black hole Farey tail*, [hep-th/0005003](#) [[SPIRES](#)].
- [25] O. Coussaert and M. Henneaux, *Supersymmetry of the $(2 + 1)$ black holes*, *Phys. Rev. Lett.* **72** (1994) 183 [[hep-th/9310194](#)] [[SPIRES](#)].
- [26] A. Strominger, *AdS_2 quantum gravity and string theory*, *JHEP* **01** (1999) 007 [[hep-th/9809027](#)] [[SPIRES](#)].
- [27] A. Sen, *Black hole entropy function, attractors and precision counting of microstates*, *Gen. Rel. Grav.* **40** (2008) 2249 [[arXiv:0708.1270](#)] [[SPIRES](#)].

- [28] V.A. Gritsenko and V.V. Nikulin, *Igusa modular forms and “the simplest” Lorentzian Kac-Moody algebras*, *Mat. Sb.* **187** (1996) 27 [[alg-geom/9603010](#)] [[SPIRES](#)].
- [29] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Birkhäuser, Germany (1985).
- [30] A. Dabholkar and D. Gaiotto, *Spectrum of CHL dyons from genus-two partition function*, *JHEP* **12** (2007) 087 [[hep-th/0612011](#)] [[SPIRES](#)].
- [31] A. Castro and S. Murthy, *Corrections to the statistical entropy of five dimensional black holes*, *JHEP* **06** (2009) 024 [[arXiv:0807.0237](#)] [[SPIRES](#)].
- [32] N. Banerjee, *Subleading correction to statistical entropy for BMPV black hole*, [arXiv:0807.1314](#) [[SPIRES](#)].
- [33] D. Zagier, *The 1-2-3 of modular forms*, Universitext, Springer, Berlin Germany (2008).
- [34] G. van der Geer, *Hilbert modular surfaces*, volume 16 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer-Verlag, Berlin Germany (1988).
- [35] J.A. Harvey and G.W. Moore, *Fivebrane instantons and R^2 couplings in $N = 4$ string theory*, *Phys. Rev. D* **57** (1998) 2323 [[hep-th/9610237](#)] [[SPIRES](#)].
- [36] A. Gregori et al., *R^2 corrections and non-perturbative dualities of $N = 4$ string ground states*, *Nucl. Phys. B* **510** (1998) 423 [[hep-th/9708062](#)] [[SPIRES](#)].
- [37] A. Sen, *Walls of Marginal Stability and Dyon Spectrum in $N = 4$ Supersymmetric String Theories*, *JHEP* **05** (2007) 039 [[hep-th/0702141](#)] [[SPIRES](#)].
- [38] S.P. Zwegers, *Mock theta functions*, Ph.D. thesis, Universiteit Utrecht, Utrecht, The Netherlands (2002).
- [39] D. Zagier, *Ramanujan’s mock theta functions and their applications*, d’après Zwegers and Bringmann-Ono, *Séminaire Bourbaki*, 60^{ème} année **986** (2007).
- [40] A. Dabholkar, S. Murthy and D. Zagier, *Quantum black holes and mock modular forms*, in preparation.
- [41] A. Sen, *Black Hole Entropy Function and the Attractor Mechanism in Higher Derivative Gravity*, *JHEP* **09** (2005) 038 [[hep-th/0506177](#)] [[SPIRES](#)].